

Morphing and barycenters

P. Lapôte

Lycée Sophie Berthelot, Calais
IREM de Lille, France

Régionale des Pays-Bas, 2011

Morphing is a special effect in motion pictures and animations that changes (or morphs) one image into another through a seamless transition. Most often it is used to depict one person turning into another through technological means or as part of a fantasy or surreal sequence. Traditionally such a depiction would be achieved through cross-fading techniques on film. Since the early 1990s, this has been replaced by computer software to create more realistic transitions.

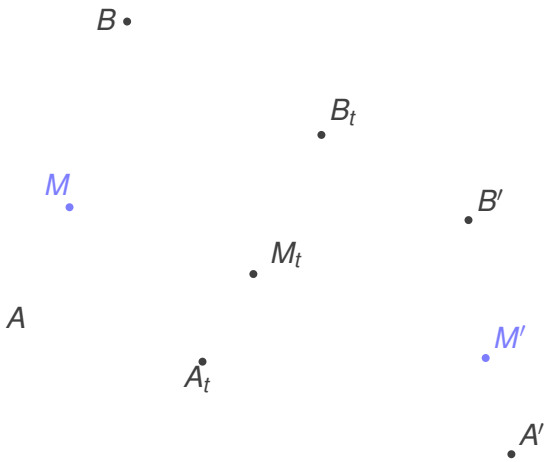
- http://youtube.com/watch?v=nUDIoN-_Hxs

Curves' morphings

From now on we shall leave pictures' morphing and deal exclusively with curves's morphing :

- segments's morphing
- triangles's morphing
- parallelograms's morphing
- squares's morphing
- nice curves's morphing

Principle of morphing



The curve AB is the source, the curve $A'B'$ is the target.
 t is the morphing parameter.

When M_θ runs along the curve AB and M' along the curve $A'B'$,
 M_t runs along the intermediate curve A_tB_t .

When t varies from 0 to 1, the curve A_tB_t goes from the curve
 AB to the curve $A'B'$.

In the following cases, θ is a barycental parameter.

Two points' barycenter

- We say that G is a barycenter of two given points A and B with respective given masses a and b if

$$a\overrightarrow{GA} + b\overrightarrow{GB} = \vec{0}$$

$$a\overrightarrow{GA} + b\overrightarrow{GB} = \vec{0} \iff (a+b)\overrightarrow{AG} = b\overrightarrow{AB}$$
$$\iff a\overrightarrow{OA} + b\overrightarrow{OB} = (a+b)\overrightarrow{OG}$$

whatever be the point O.

If $a + b \neq 0$,

$$a\overrightarrow{GA} + b\overrightarrow{GB} = \vec{0} \iff \overrightarrow{AG} = \frac{b}{a+b}\overrightarrow{AB}$$

$\implies G$ is unique

Barycenter of three or more points

Let A , B and C be three points
with respective masses a , b and c .

G is a barycenter of (A, a) , (B, b) , (C, c) if

$$a\overrightarrow{GA} + b\overrightarrow{GB} + c\overrightarrow{GC} = \vec{0}$$

Barycenter of three or more points

Whatever be the point O

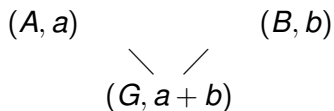
$$a\overrightarrow{GA} + b\overrightarrow{GB} + c\overrightarrow{GC} = \vec{0}$$
$$\iff (a + b + c)\overrightarrow{OG} = a\overrightarrow{OA} + b\overrightarrow{OB} + c\overrightarrow{OC}$$

If $a + b + c \neq 0$, G is unique.

And so on ...

Elementary barycenter's properties

- The following diagram



- has to be understood as
- G is a barycenter of $(A, a), (B, b)$

Elementary barycenter's properties

- The following diagram

$$\begin{array}{ccc} (A, a) & & (B, b) \\ & \diagdown \quad \diagup & \\ & (G, a + b) & \end{array}$$

- has to be understood as
- G is a barycenter of $(A, a), (B, b)$

Elementary barycenter's properties

- The following diagram

$$\begin{array}{ccc} (A, a) & & (B, b) \\ & \searrow \quad \swarrow & \\ & (G, a + b) & \end{array}$$

- has to be understood as
- G is a barycenter of $(A, a), (B, b)$

Exchangeability

$$\begin{array}{ccc} (A, a) & & (B, b) \\ & \searrow \quad \swarrow & \\ & (G, a + b) & \\ & \swarrow \quad \searrow & \\ (B, b) & & (A, a) \end{array} \iff \begin{array}{ccc} (B, b) & & (A, a) \\ & \searrow \quad \swarrow & \\ & (G, a + b) & \\ & \swarrow \quad \searrow & \\ (A, a) & & (B, b) \end{array}$$

Exchangeability

$$\begin{array}{ccc} (A, a) & & (B, b) \\ & \diagdown \quad \diagup & \\ & (G, a + b) & \\ & & \Leftrightarrow \\ & & (G, a + b) & & (B, -b) \\ & & & \diagdown \quad \diagup & \\ & & & (A, a) & \end{array}$$

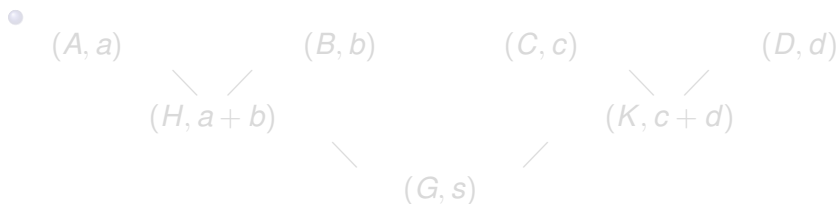
Homogeneity

For every real number $\lambda \neq 0$,

$$\begin{array}{ccc} (A, a) & & (B, b) \\ & \diagdown \quad \diagup & \\ & (G, a + b) & \end{array} \iff \begin{array}{ccc} (A, \lambda a) & & (B, \lambda b) \\ & \diagdown \quad \diagup & \\ & (G, \lambda(a + b)) & \end{array}$$

Barycenter's associativity : example

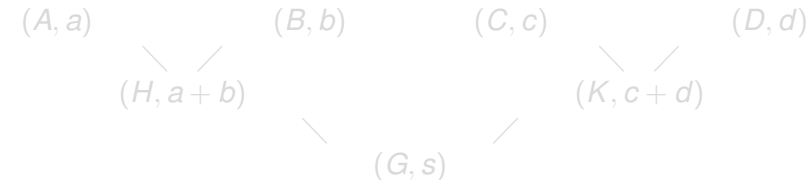
- G is a barycenter of $(A, a), (B, b), (C, c), (D, d)$ if and only if it is a barycenter of $(H, a + b), (K, c + d)$
- where H is a barycenter of $(A, a), (B, b)$ and K is a barycenter of $(C, c), (D, d)$.



where $s = a + b + c + d$

Barycenter's associativity : example

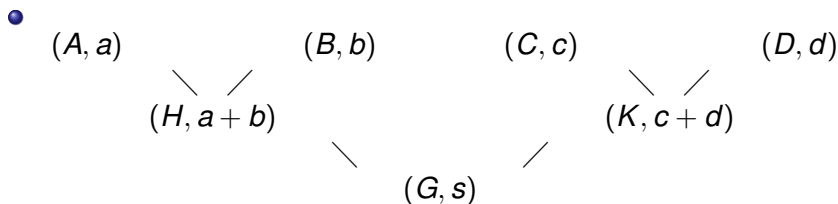
- G is a barycenter of $(A, a), (B, b), (C, c), (D, d)$ if and only if it is a barycenter of $(H, a + b), (K, c + d)$
- where H is a barycenter of $(A, a), (B, b)$ and K is a barycenter of $(C, c), (D, d)$.



where $s = a + b + c + d$

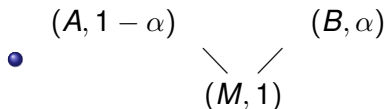
Barycenter's associativity : example

- G is a barycenter of $(A, a), (B, b), (C, c), (D, d)$ if and only if it is a barycenter of $(H, a + b), (K, c + d)$
- where H is a barycenter of $(A, a), (B, b)$ and K is a barycenter of $(C, c), (D, d)$.



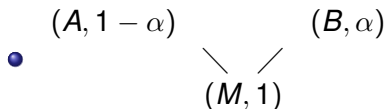
where $s = a + b + c + d$

Let



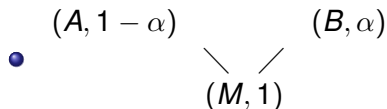
- When α increases from 0 to 1, M runs along $[AB]$, from A to B .
- $\overrightarrow{AM} = \alpha \overrightarrow{AB}$

Let



- When α increases from 0 to 1, M runs along $[AB]$, from A to B .
- $\overrightarrow{AM} = \alpha \overrightarrow{AB}$

Let



- When α increases from 0 to 1, M runs along $[AB]$, from A to B .
- $\overrightarrow{AM} = \alpha \overrightarrow{AB}$

Segments's morphing : $[AB] \longrightarrow [A'B']$

- Let $[AB]$ and $[A', B']$ two segments.

- Assume $0 \leq \theta \leq 1$

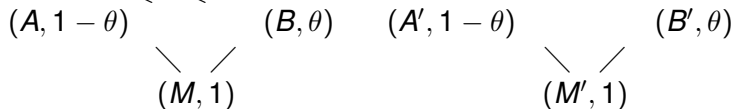


- M is the generic point of $[AB]$, M' the generic point of $[A'B']$
- Both for $[AB]$ and $[A'B']$, θ is the usual parameter related to barycenters ; this is a choice among many other choices of parameters.

Segments's morphing : $[AB] \longrightarrow [A'B']$

- Let $[AB]$ and $[A', B']$ two segments.

- Assume $0 \leq \theta \leq 1$



- M is the generic point of $[AB]$, M' the generic point of $[A'B']$
- Both for $[AB]$ and $[A'B']$, θ is the usual parameter related to barycenters ; this is a choice among many other choices of parameters.

Segments's morphing : $[AB] \longrightarrow [A'B']$

- Let $[AB]$ and $[A', B']$ two segments.

- Assume $0 \leq \theta \leq 1$

$$\begin{array}{ccccccc} (A, 1 - \theta) & & (B, \theta) & & (A', 1 - \theta) & & (B', \theta) \\ & \searrow \quad \swarrow & & & \searrow \quad \swarrow & & \\ & (M, 1) & & & (M', 1) & & \end{array}$$

- M is the generic point of $[AB]$, M' the generic point of $[A'B']$
- Both for $[AB]$ and $[A'B']$, θ is the usual parameter related to barycenters ; this is a choice among many other choices of parameters.

Segments's morphing : $[AB] \longrightarrow [A'B']$

- Let $[AB]$ and $[A', B']$ two segments.

- Assume $0 \leq \theta \leq 1$

$$\begin{array}{ccccccc} (A, 1 - \theta) & & (B, \theta) & & (A', 1 - \theta) & & (B', \theta) \\ & \searrow & / & & \searrow & / & \\ & (M, 1) & & & (M', 1) & & \end{array}$$

- M is the generic point of $[AB]$, M' the generic point of $[A'B']$
- Both for $[AB]$ and $[A'B']$, θ is the usual parameter related to barycenters ; this is a choice among many other choices of parameters.

Segments's morphing : $[AB] \longrightarrow [A'B']$

- Morphing : let t , $0 \leq t \leq 1$ be the morphing parameter.

$$\begin{array}{ccccc} (A, 1-t) & & (A', t) & & (B, 1-t) & & (B', t) \\ & \searrow \swarrow & & & \searrow \swarrow & & \\ & (A_t, 1) & & & (B_t, 1) & & \end{array}$$

$$\begin{array}{ccc} (M, 1-t) & & (M', t) \\ & \searrow \swarrow & \\ & (M_t, 1) & \end{array}$$

Segments's morphing : $[AB] \longrightarrow [A'B']$

- The locus of M_t for fixed t when θ increases from 0 to 1 is the moving image of $[AB]$ during the morphing when t varies.
- For $t = 0$, it is $[AB]$; for $t = 1$, it is $[A'B']$.
- For every $t \in] 0, 1[$, it is a segment.

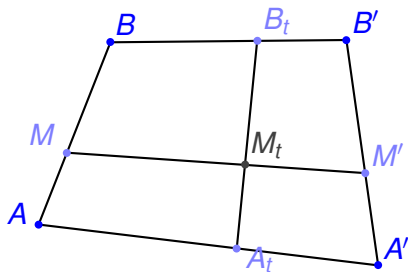
Segments's morphing : $[AB] \longrightarrow [A'B']$

- The locus of M_t for fixed t when θ increases from 0 to 1 is the moving image of $[AB]$ during the morphing when t varies.
- For $t = 0$, it is $[AB]$; for $t = 1$, it is $[A'B']$.
- For every $t \in] 0, 1[$, it is a segment.

Segments's morphing : $[AB] \longrightarrow [A'B']$

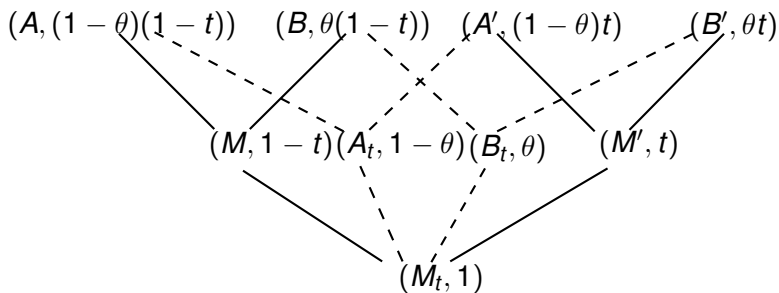
- The locus of M_t for fixed t when θ increases from 0 to 1 is the moving image of $[AB]$ during the morphing when t varies.
- For $t = 0$, it is $[AB]$; for $t = 1$, it is $[A'B']$.
- For every $t \in] 0, 1[$, it is a segment.

Illustration



We now prove that the running curve is a segment

Proof



Consequences


- $\implies \overrightarrow{A_t M_t} = \theta \overrightarrow{A_t B_t}$
- $M_t \in [A_t B_t]$

Consequences

- $\implies \overrightarrow{A_t M_t} = \theta \overrightarrow{A_t B_t}$
- $M_t \in [A_t B_t]$

Parallelograms's morphing

A morphing from a parallelogram $ABCD$ to a parallelogram $A'B'C'D'$ produces a moving parallelogram.

- Let $ABCD$ and $A'B'C'D'$ be parallelograms
- Let us define C_t and D_t like A_t and B_t previously
 $(C, 1-t)$ (C', t) $(D, 1-t)$ (D', t)

 $(C_t, 1)$ $(D_t, 1)$
- then, $A_t B_t C_t D_t$ is still a parallelogram.

Parallelograms's morphing

A morphing from a parallelogram $ABCD$ to a parallelogram $A'B'C'D'$ produces a moving parallelogram.

- Let $ABCD$ and $A'B'C'D'$ be parallelograms

- Let us define C_t and D_t like A_t and B_t previously

$$\begin{array}{ccccccc} (C, 1-t) & & (C', t) & & (D, 1-t) & & (D', t) \\ & \searrow \swarrow & & & \searrow \swarrow & & \\ & (C_t, 1) & & & (D_t, 1) & & \end{array}$$

- then, $A_t B_t C_t D_t$ is still a parallelogram.

Parallelograms's morphing

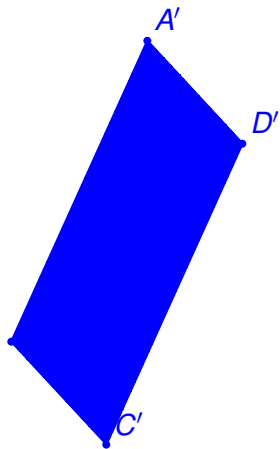
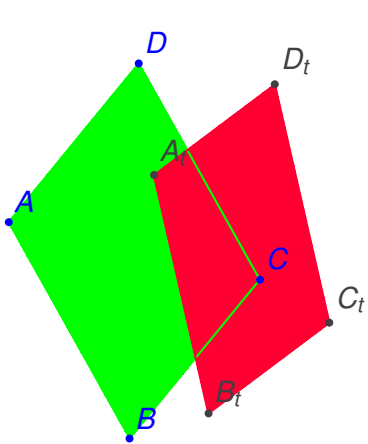
A morphing from a parallelogram $ABCD$ to a parallelogram $A'B'C'D'$ produces a moving parallelogram.

- Let $ABCD$ and $A'B'C'D'$ be parallelograms

- Let us define C_t and D_t like A_t and B_t previously

$$\begin{array}{ccccccc} (C, 1-t) & & (C', t) & & (D, 1-t) & & (D', t) \\ & \searrow \swarrow & & & \searrow \swarrow & & \\ & (C_t, 1) & & & (D_t, 1) & & \end{array}$$

- then, $A_t B_t C_t D_t$ is still a parallelogram.



- one remark :

$ABCD$ parallelogram

is equivalent to

$$\overrightarrow{AB} = \overrightarrow{DC}$$

is equivalent to

$$\overrightarrow{AB} = \overrightarrow{DA} + \overrightarrow{AC}$$

is equivalent to

$$\overrightarrow{AB} - \overrightarrow{AC} + \overrightarrow{AD} = \vec{0}$$

is equivalent to

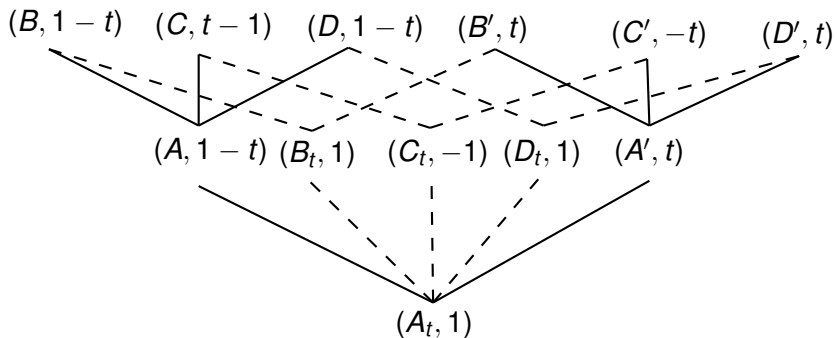
$$A = \text{bar}\{(B; 1), (C; -1), (D; 1)\}$$

By homogeneity

$$\begin{array}{ccccc} (B, 1) & (C, -1) & (D, 1) & (B', 1) & (C', -1) & (D', 1) \\ & \searrow \downarrow \swarrow & & & \searrow \downarrow \swarrow & \\ & (A, 1) & & & (A', 1) & \end{array}$$

$$\begin{array}{ccccc} (B, 1 - t) & (C, t - 1) & (D, 1 - t) & (B', t) & (C', -t) & (D', t) \\ & \searrow \downarrow \swarrow & & & \searrow \downarrow \swarrow & \\ & (A, 1 - t) & & & (A', t) & \end{array}$$

By associativity



- five points -> five points morphing
- segment -> segment morphing (with non barycentral parameters)
- segment -> segment morphing
- quadrilateral -> quadrilateral morphing
- square -> circle morphing
- heart -> diamond morphing

- square-circle's morphing
- hearts-circle's morphing
- four petals's flowers-circle's morphing
- apple-four petals's flower's morphing

- Louis-Philippe's caricature